Asymptotic Analysis

- Runtime Analysis and Big Oh - $O(\cdot)$
- Complexity Classes and Curse of Exponential Time
- $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$ - Relational properties

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Comparing Algorithms

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Which one has smaller runtime?

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Can we compare functions? $n^2 + 2n \ < \ 3n^2$

Function: List Representation

Let X and Y be two sets. A function f maps **each** element of X to **exactly one** element of Y

$f : X \mapsto Y$

- X is the domain of f
- Y is the codomain of f

$f(x) = y$

- y is the image of x
- x is the pre-image of y

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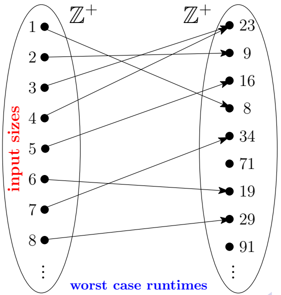
Worstcase runtime of algorithms characterized as functions of input sizes



input sizes

1	2	3	4	5	6	7	8	...
8	9	23	23	16	19	34	29	...

worst case runtimes

Asymptotic Notation

- We use asymptotic analysis of functions to analyze algorithms running time
- Characterize running time for all inputs instances of a certain size (so worst-case) with just one runtime function
- Small inputs are not much of a problem, we want to learn behavior of an algorithm on large inputs

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Goal 2: Determine how the runtime **grows with increasing inputs**

▷ How the runtime changes when input size is doubled/tripled?

Asymptotic Analysis: Big Oh

Definition (Big Oh)

A function $g(n) \in O\big(f(n)\big)$ if there exists constants $c > 0$ and $n_0 \geq 0$ such that

$$\forall n \geq n_0 \qquad g(n) \leq c \cdot f(n)$$

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Provides the right framework for both our goals

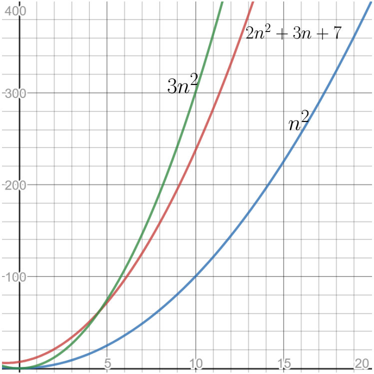
Big Oh

A function $g(n) \in O\big(f(n)\big)$ if there exists constants $c > 0$ and $n_0 \geq 0$ such that

$$\forall n \geq n_0 \qquad g(n) \leq c \cdot f(n)$$

$$2n^2 + 3n + 7 \;=\; O(n^2)$$

▷ $c = 3$ and $n_0 = 5$



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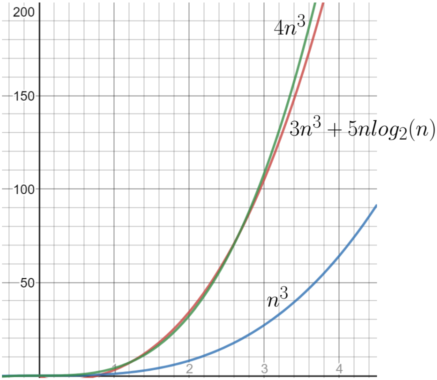
$3n^3 + 5n\log n = O(n^3)$

▷ $c = 4$ and $n_0 = 3$



Big Oh: Common Rules

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- Lower order terms are ignored
  - n^a dominates n^b if $a > b$
- Multiplicative constants are omitted
  - e.g. $7n^4 + 3n^3 + 10 \; = \; O(n^4)$
  - e.g. $3n^3 + 5n \log n \; = \; O(n^3)$

Big Oh: Common Rules

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$$\forall n \geq n_0 \qquad g(n) \leq c \cdot f(n)$$

$$\begin{aligned} f(n) &= pn^2 + qn + r \\ &\leq |p|n^2 + |q|n^2 + |r|n^2 \\ &= \big(|p| + |q| + |r|\big)n^2 \end{aligned}$$

This is true for all $n \geq 1$, hence with $c = \big(|p| + |q| + |r|\big)$ we get that $f(n) = O(n^2)$

Big Oh: Justification to ignore lower order terms

Let the runtime of algorithm $\mathcal{A}$ be $\quad T(n) := n^2 + 10n$

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Consider an input size of 10^9, then

$$n^2 + 100n \;=\; 10^{18} + 10^{11} \qquad \text{and} \qquad n^2 \;=\; 10^{18}$$

$$\text{fractional error} \;=\; \frac{10^{11}}{10^{18}} \;=\; 10^{-7}$$

Goal 1: Determine the runtime of an algorithm on inputs of **large size**

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For $n = 10^9,\ T(n) = n^2 + 10n$ is only 0.00001% more than n^2

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Coefficients do not really affect growth of functions

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Linear $f(n) = 5n$

$$\begin{aligned} n &: 5n \\ 2n &: 2(5n) \\ 3n &: 3(5n) \\ 4n &: 4(5n) \end{aligned}$$

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Quadratic $f(n) = 7n^2$

$$\begin{aligned} n &: 7n^2 \\ 2n &: 4(7n^2) \\ 3n &: 9(7n^2) \\ 4n &: 16(7n^2) \end{aligned}$$

Big Oh: Justification to ignore Coefficients

Coefficients do not really affect growth of functions

Goal 2: Determine how the runtime **grows with increasing inputs**

Linear $f(n) = 5n$	Quadratic $f(n) = 7n^2$	Cubic $f(n) = 2n^3$
$n : 5n$	$n : 7n^2$	$n : 2n^3$
$2n : 2(5n)$	$2n : 4(7n^2)$	$2n : 8(2n^3)$
$3n : 3(5n)$	$3n : 9(7n^2)$	$3n : 27(2n^3)$
$4n : 4(5n)$	$4n : 16(7n^2)$	$4n : 64(2n^3)$

Since we are concerned with scalability of algorithm, the same growth factor is observed if we consider $f(n) = n$, $f(n) = n^2$, and $f(n) = n^3$

Non tightness of Big Oh

Note: $O(\cdot)$ expresses only an upper bound on growth rate of a function

▷ It does not necessarily give the exact growth rate of the function

For example, $f(n) = 3n^2 + 4n + 5 \ = \ O(n^2)$ it is also $O(n^3)$

Indeed, $f(n) \ \leq \ 12n^2$ and also $f(n) \ \leq \ 12n^3$

Big Oh: Finding the right constants

Let $g(n) \;=\; 7n+4$ and $f(n) \;=\; n$

$g(n) = O(f(n))$

$\triangleright$ take $c = 8$ and $n_0 = 4$, $7n+4 \leq 8(n)$ whenever $n_0 \geq 4$,

1. To get c: We want $7n+4 \leq cn$

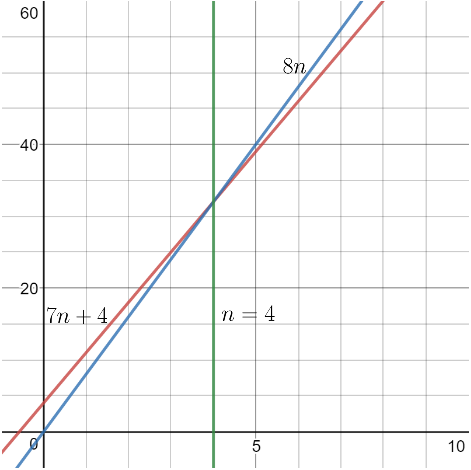
   Solving the inequality for c, we get $c \geq \frac{7n}{n} + \frac{4}{n}$

   For $n \geq 4$, $8 \geq 7 + \frac{4}{n}$

   $\triangleright$ Observe that $\lim_{n\to\infty} \frac{7n+4}{n} \to 7$, but this $(c = 7)$ would require n_0 to be approaching ∞, so we take $c = 8$

2. To get n_0: We want $7n+4 \leq 8n$, this is true whenever $n \geq 4$

Big Oh: Finding the right constants



Big Oh: Finding the right constants

Let $f(n) \ = \ 6n + 24$ and $h(n) = n^2$,

$$f(n) \ = \ O(h(n))$$

As $\lim_{n\to\infty} \dfrac{6n+24}{n^2} \to 0$, so any $c > 0$ will work.

for $c = 1$, we want

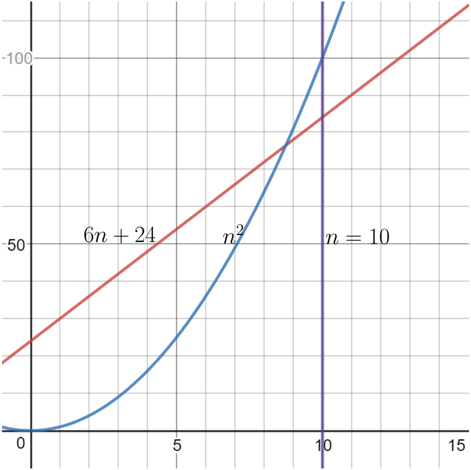
$6n + 24 \leq 1 \cdot n^2$

which is true whenever $n \geq 10$

So we choose $c = 1$ and $n_0 = 10$

More examples in lecture notes and problem set

Big Oh: Finding the right constants

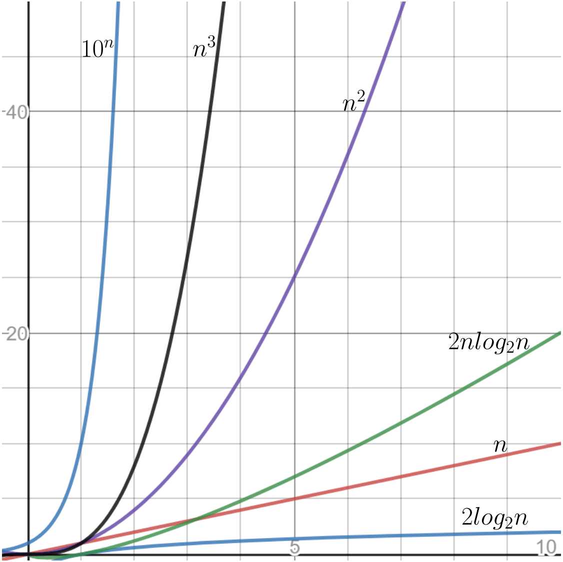


Asymptotic-Complexity Classes

Class Name	**Class Symbol**	**Example**
Constant	$O(1)$	Comparison of two integers
Logarithmic	$O(log(n))$	Binary Search, Exponentiation
Linear	$O(n)$	Linear Search
Log-Linear	$On(log(n))$	Merge Sort
Quadratic	$O(n^2)$	Integer multiplications
Cubic	$O(n^3)$	Matrix multiplication
Polynomial	$O(n^a)$, $a \in \mathbb{R}$	
Exponential	$O(a^n)$, $a \in \mathbb{R}$	Print all subsets
Factorial	$O(n!)$	Print all permutations

$$n! \gg 2^n \gg n^3 \gg n^2 \gg nlogn \gg n \gg logn \gg 1$$

Growth Rates of Functions



Big Oh: Why does it make sense?

Runtimes of algorithms of different runtime for input size n (on 1GHz PC). Assume that each operation takes 1 ns

n	$O(\log n)$	$O(n)$	$O(n \log n)$	$O(n^2)$	$O(2^n)$	$O(n!)$
10	$0.003\mu s$	$0.01\mu s$	$0.033\mu s$	$0.1\mu s$	$1\mu s$	$3.63 ms$
20	$0.004\mu s$	$0.02\mu s$	$0.086\mu s$	$0.4\mu s$	$1 ms$	77.1 yrs
30	$0.005\mu s$	$0.03\mu s$	$0.147\mu s$	$0.9\mu s$	$1 sec$	$8 \cdot 10^{15}$ yrs
40	$0.005\mu s$	$0.04\mu s$	$0.213\mu s$	$1.6\mu s$	$18.3 min$	very long
50	$0.006\mu s$	$0.05\mu s$	$0.282\mu s$	$2.5\mu s$	13 $days$	very long
100	$0.007\mu s$	$0.10\mu s$	$0.644\mu s$	$10\mu s$	$4 \cdot 10^{13}$ yrs	very long
10^3	$0.010\mu s$	$1.00\mu s$	$9.966\mu s$	$1 ms$	very long	very long
10^4	$0.013\mu s$	$10\mu s$	$130\mu s$	$100 ms$	very long	very long
10^5	$0.017\mu s$	$0.10 ms$	$1.67 ms$	$10 sec$	very long	very long
10^6	$0.020\mu s$	$1 ms$	$19.93 ms$	$16.7 min$	very long	very long
10^7	$0.023\mu s$	$0.01 sec$	$0.23 sec$	1.16 $days$	very long	very long
10^8	$0.027\mu s$	$0.10 sec$	$2.66 sec$	115.7 $days$	very long	very long
10^9	$0.030\mu s$	$1 sec$	$29.90 sec$	31.7 yrs	very long	very long